Dynamic Structural Causal Models

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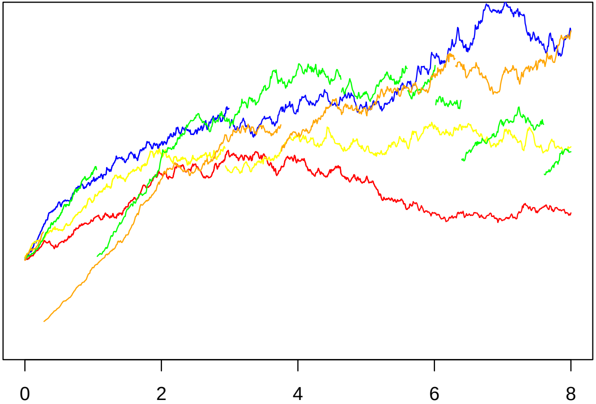
Continuous-time Dynamical Systems



$$X(t) = X^0 + \int_0^t f_1(X(s), Y(s))\mathrm{d}s + \int_0^t f_2(X(s), Y(s))\mathrm{d}W(s) + \int_0^t f_3(X(s-), Y(s-))\mathrm{d}N(s)$$

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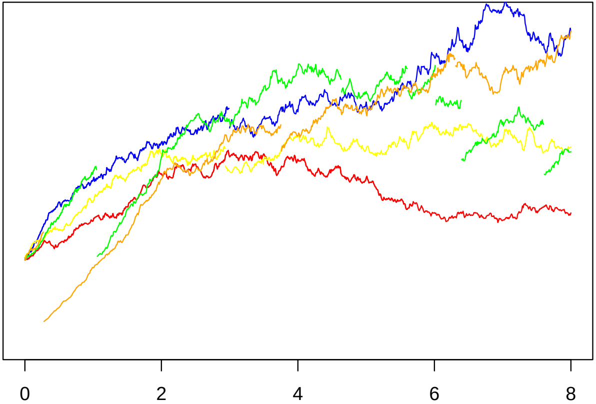
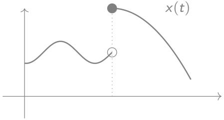


$$\left.\begin{matrix} x^5 \\ x^4 \\ x^3 \\ x^2 \\ x^1 \end{matrix}\right\} \in D_{\mathcal{T},\mathbb{R}}$$

functions
$x : \mathcal{T} \to \mathbb{R}$
right continuous
with left limits



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Dynamic Structural Causal Models

Structural Causal Model:

- Endogenous V, exogenous W
- Outcome spaces $\mathcal{X}_V$
- Structural equations $X_V = f_v(X_V, X_W)$
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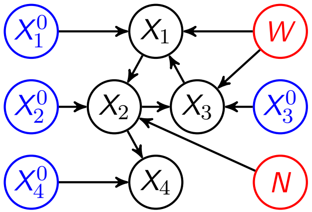
Static SCM	
State space	$\mathcal{X}_v \subseteq \mathbb{R}$
Variable	$X_v \in \mathbb{R}$
Structural equation	$f_v : \mathbb{R}^{m+n} \to \mathbb{R}$

Dynamic SCM	
Function space	$\mathcal{X}_v = D_{\mathcal{T},\mathbb{R}}$
Trajectory	$X_v \in D_{\mathcal{T},\mathbb{R}}$ or $X_v : \mathcal{T} \to \mathbb{R}$
Structural equation	$f_v : D^{m+n}_{\mathcal{T},\mathbb{R}} \to D_{\mathcal{T},\mathbb{R}}$

From a Structural System of SDEs to a DSCM

$$\mathcal{D}: \begin{cases} X_1(t) = X_1^0 + \int_0^t g_1(s-, X_1, X_3)\mathrm{d}W(s) \\ X_2(t) = X_2^0 + \int_0^t g_2(s-, X_1, X_2)\mathrm{d}N(s) \\ X_3(t) = X_3^0 + \int_0^t g_3(s-, X_2, X_3)\mathrm{d}W(s) \\ X_4(t) = X_4^0 + \int_0^t g_4(s-, X_4)\mathrm{d}X_2(s) \end{cases}$$



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From a Structural System of SDEs to a DSCM

Using Przybyłowicz et al. (2023)

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Description of dynamical evolution of states

Functional dependencies of trajectories

From a Structural System of SDEs to a DSCM

Using Przybyłowicz et al. (2023)

$$\xrightarrow{\hspace{2cm}}$$

$$\mathcal{D}: \begin{cases} X_1(t) = X_1^0 + \int_0^t g_1(s-, X_1, X_3)\mathrm{d}W(s) \\ X_2(t) = X_2^0 + \int_0^t g_2(s-, X_1, X_2)\mathrm{d}N(s) \\ X_3(t) = X_3^0 + \int_0^t g_3(s-, X_2, X_3)\mathrm{d}W(s) \\ X_4(t) = X_4^0 + \int_0^t g_4(s-, X_4)\mathrm{d}X_2(s) \end{cases}$$

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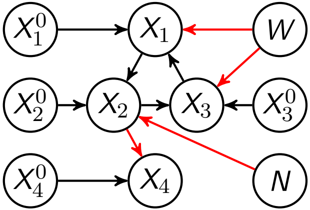
From a Structural System of SDEs to a DSCM

$$
\begin{tikzcd}
\mathcal{D}: \begin{cases} X_1(t) = X_1^0 + \int_0^t g_1(s-, X_1, X_3)\,\mathrm{d}W(s) \\ X_2(t) = X_2^0 + \int_0^t g_2(s-, X_1, X_2)\,\mathrm{d}N(s) \\ X_3(t) = X_3^0 + \int_0^t g_3(s-, X_2, X_3)\,\mathrm{d}W(s) \\ X_4(t) = X_4^0 + \int_0^t g_4(s-, X_4)\,\mathrm{d}X_2(s) \end{cases} \arrow[r, "\text{Using Przybyłowicz et al. (2023)}"] \arrow[d] & \mathcal{M}_{\mathcal{D}}: \begin{cases} X_1 = f_1(X_1^0, X_3, W) \\ X_2 = f_2(X_2^0, X_1, N) \\ X_3 = f_3(X_3^0, X_2, W) \\ X_4 = f_4(X_4^0, X_2) \end{cases} \arrow[d] \\
\mathcal{D}_{\mathrm{do}(X_3 = x_3)} \arrow[r] & \begin{array}{c} (\mathcal{M}_{\mathcal{D}})_{\mathrm{do}(X_3 = x_3)} \\ = \\ \mathcal{M}_{(\mathcal{D}_{\mathrm{do}(X_3 = x_3)})} \end{array}
\end{tikzcd}
$$

DSCM Markov Property

A DSCM is a genuine (cyclic) SCM as in Bongers et al. (2021).

Graph $G(\mathcal{M})$:



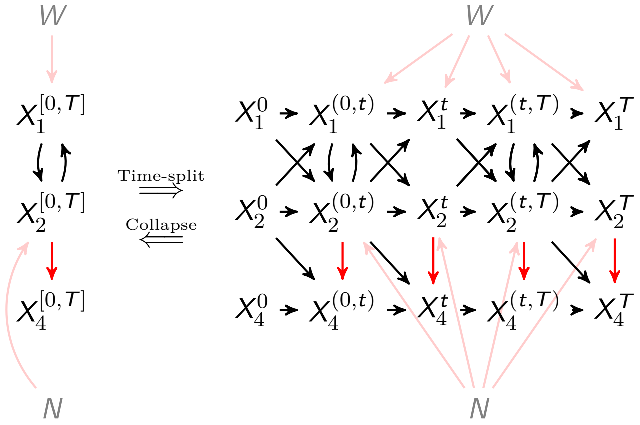
Markov property (Forré and Mooij, 2017):

$$X_1 \overset{\sigma}{\underset{G}{\perp}} X_4 \mid X_2 \implies X_1 \underset{\mathbb{P}(X_V)}{\perp\!\!\!\perp} X_4 \mid X_2$$

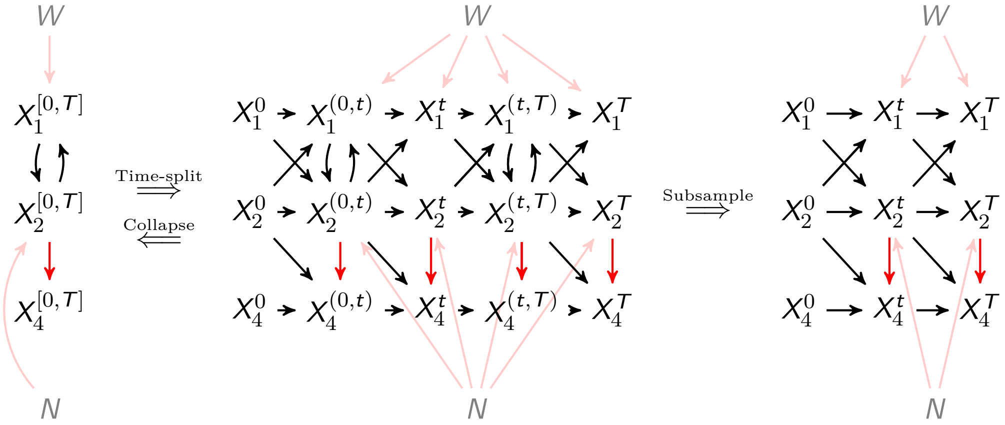
Time-splitting, Subsampling

$$
\begin{tikzcd}
W \arrow[d] \\
X_1^{[0,T]} \arrow[d, shift right] \\
X_2^{[0,T]} \arrow[u, shift right] \arrow[d] \\
X_4^{[0,T]} \\
N \arrow[uu, bend left]
\end{tikzcd}
$$

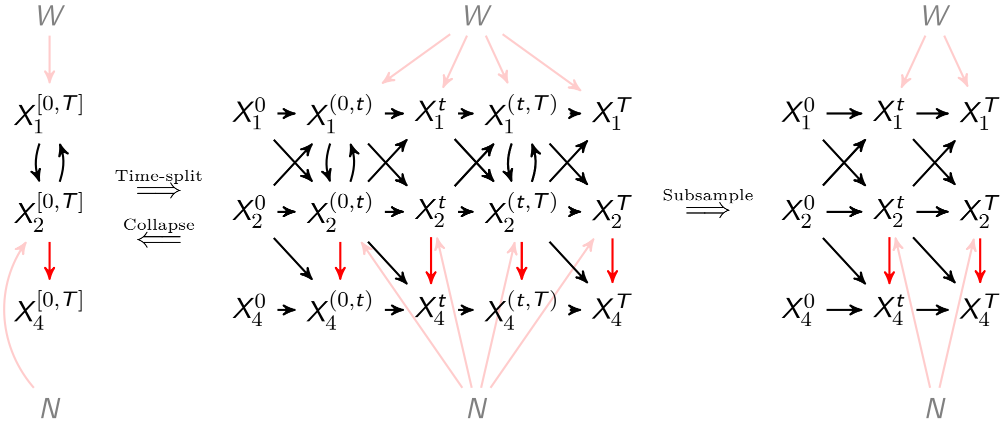
Time-splitting, Subsampling



Time-splitting, Subsampling



Time-splitting, Subsampling



Time-splitting is invertible, but subsampling is not!

Results for DSCMs

We can apply existing ‘static’ SCM methods to dynamical systems:

Cyclic causal effect identification:

- Do-calculus (Forré and Mooij, 2020)
- Generalised adjustment formulae (Forré and Mooij, 2020)
- ID algorithm (Forré and Mooij, 2023)

Cyclic constraint-based causal discovery:

- LCD (Cooper, 1997; Forré and Mooij, 2023)
- Y-structures (Mani, 2006; Forré and Mooij, 2023)
- FCI (Spirtes et al., 1995; Forré and Mooij, 2023)

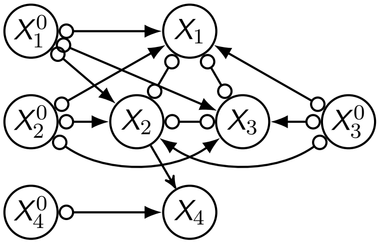
Causal Effect Identification: Backdoor Adjustment

$$\begin{tikzcd}
& W \arrow[dl] \arrow[dr] & \\
X_1^{[0,T]} \arrow[rr, bend left] & & X_2^{[0,T]} \arrow[ll, bend left]
\end{tikzcd}$$

$$\mathbb{P}(X_2|\,\mathrm{do}(X_1 = x)) = \int_{D_{\mathcal{T},\mathbb{R}}} \mathbb{P}(X_2|X_1 = x, W = w)\mathrm{d}\mathbb{P}(w)$$

Constraint-based Causal Discovery: FCI

Output of FCI with independence oracle:



In practice, require conditional independence test for e.g. $X_1 \perp\!\!\!\perp X_4 \mid X_2$ (Lundborg et al., 2022; Manten et al., 2024).

Local Conditional Independence

Let $X_A \not\to X_B \mid X_C$ denote local conditional independence of X_B from X_A given X_C.

Theorem

For a collapsed DSCM $\mathcal{M}$ with no instantaneous effects and independent integrators, $G(\mathcal{M})$ is a local independence graph and

$$X_A \underset{G(\mathcal{M})}{\overset{\sigma}{\perp}} X_B | X_C \implies X_A \not\to X_B \mid X_C.$$

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